

College NAME - S-S-College. B.Sc Part III
 Subject - Math
 Time - 10 A.M to 12 A.M Topic - Uniform Convergence (Real analysis)
 Date - 14-05-2020 By - Shree Nalini Sharma

8. State and Prove the Weierstrass Approximation Theorem -

Statement - Let f be a continuous real function defined on closed interval $[a, b]$ and let $\epsilon > 0$ be given then there exists a polynomial p with real coefficients such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a, b]$

Proof: $\rightarrow$ STEP I $\rightarrow$

First we prove the result for closed interval $[0, 1]$

For $x \in [0, 1]$,

$$\text{we have } \sum_{k=0}^n nC_k (1-x)^{n-k} x^k = (1-x+x)^n = 1 \quad (1)$$

Differentiating (1) with respect to x , we get-

$$\sum_{k=0}^n nC_k \left[-(n-k)(1-x)^{n-k-1} x^k + k(1-x)^{n-k} x^{k-1} \right] = 0$$

$$\Rightarrow \sum_{k=0}^n nC_k x^{k-1} (1-x)^{n-k-1} (k-nx) = 0 \quad (2)$$

Multiplying this by $x(1-x)$, we get-

$$\sum_{k=0}^n nC_k x^k (1-x)^{n-k} (k-nx) = 0 \quad (3)$$

We differentiate (3) with respect to x

Considering $x^k (1-x)^{n-k}$ as one of the two factors in applying the product rule.

this using differentiation (2) of (1) we get-

$$\sum_{k=0}^n nC_k \left[x^{k+1} (1-x)^{n-k-1} (k-nx) - nx^k (1-x)^{n-k} \right] = 0$$

$$\text{i.e. } \sum_{k=0}^n nC_k x^k (1-x)^{n-k} = \sum_{k=0}^n nC_k x^k (1-x)^{n-k} = 1 \quad (4)$$

Multiplying both sides by $x(1-x)$ and dividing both sides by n^2 , we get.

$$\sum_{k=0}^n nC_k x^k (1-x)^{n-k} \left(x - \frac{k}{n}\right)^2 = \frac{x(1-x)}{n} \quad (5)$$

Now we consider Bernstein polynomial associated with f and defined for each n

$$\text{by } B_n(x) = \sum_{k=0}^n nC_k x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \quad (6)$$

We shall prove that $\{B_n(x)\}$ converges uniformly to $f(x)$

Using (1) we have

$$f(x) = \sum_{k=0}^n nC_k x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

Thus

$$\begin{aligned} |f(x) - B_n(x)| &= \left| \sum_{k=0}^n nC_k x^k (1-x)^{n-k} \left[f(x) - f\left(\frac{k}{n}\right) \right] \right| \\ &\leq \sum_{k=0}^n nC_k x^k (1-x)^{n-k} \left| f(x) - f\left(\frac{k}{n}\right) \right| \quad (7) \end{aligned}$$

Since f is continuous on $[0,1]$, it is uniformly continuous on $[0,1]$. So there exists $\delta > 0$ such that

$$\left| x - \frac{k}{n} \right| < \delta \Rightarrow \left| f(x) - f\left(\frac{k}{n}\right) \right| < \epsilon/2 \quad (8)$$

Now we split the sum on R.H.S of (7) into two parts and denote them by $\mathcal{A}$ and $\mathcal{B}$ where

Σ is the sum of those terms for which $|x - \frac{k}{n}| < \delta$ and



Σ' is the sum of ~~those~~ the remaining terms. (we consider x to be fixed for some moment).
Then from (1) and (4) we have

$$\Sigma < \epsilon/2$$

Since f is continuous it is bounded and so there exists a +ve real no K such that

$$|f(x)| \leq K \text{ for all } x \in [0,1]$$

thus we see that-

$$\Sigma' \leq 2K \Sigma''$$

$$\text{where } \Sigma'' = \sum n C_k x^k (1-x)^{n-k}$$

is taken over all k for which

$$|x - \frac{k}{n}| \geq \delta,$$

thus

$$\delta^2 \Sigma'' \leq \sum n C_k x^k (1-x)^{n-k} \left(x - \frac{k}{n}\right)^2 \leq \frac{x(1-x)}{n}$$

$$\text{i.e. } \Sigma'' \leq \frac{x(1-x)}{n\delta^2}$$

The maximum value of $x(1-x)$

on $[0,1]$ is $\frac{1}{4}$. so

$$\Sigma'' \leq \frac{1}{4\delta^2 n}$$

If we choose $n_0 > \frac{K}{\delta^2 \epsilon}$ then $\Sigma'' \leq \frac{\epsilon}{4K}$ for all $n \geq n_0$

thus $\Sigma' < \epsilon/2$.

Hence $|f(x) - B_n(x)| < \epsilon$ for $n \geq n_0$.

and for all $x \in [0,1]$.

If we assume $p(x) = B_n(x)$ for $n \geq n_0$.

we get- $|f(x) - p(x)| < \epsilon$ for all $x \in [0,1]$

Step II - Now we prove the result for any closed $[a, b]$.

If $a = b$, then the result is true for the constant polynomial $p(x) = f(a)$.
So we assume that $a < b$.

Since $x \mapsto [(b-a)x' + a]$ is a ~~continuous~~ continuous function defined on $[0, 1]$ to $[a, b]$,
the function g defined by

$$g(x') =$$

So applying result of Step I we get polynomial p' such that

$$|g(x') - p'(x')| < \epsilon \text{ for all } x' \in [0, 1]$$

This inequality may also be written as

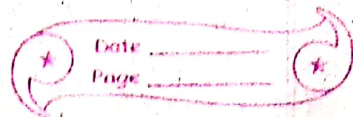
$$\left| f(x) - p' \frac{x-a}{b-a} \right| < \epsilon \text{ for all } x \in [a, b]$$

If we put $p(x) = p' \frac{x-a}{b-a}$ then

$$|f(x) - p(x)| < \epsilon \text{ for all } x \in [a, b]$$

Hence the result

Solved Ex :-



1. Show that the sequence $\{f_n\}$ of functions where

$$f_n(x) = \frac{x}{x+n}, \text{ is uniformly convergent.}$$

in $[0, k]$ where k is finite but not uniformly convergent in $[0, \infty]$

Solution $\rightarrow$ The sequence $\{f_n\}$ is point-wise convergent for all $x \geq 0$ and the point-wise limit f is

$$\begin{aligned} f(x) &= \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x}{x+n} = \lim_{n \rightarrow \infty} \frac{x}{n(\frac{x}{n}+1)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\frac{x}{n}+1} = 1 \end{aligned}$$

Let $\epsilon > 0$ be given, we have

$$|f_n(x) - f(x)| = \left| \frac{x}{x+n} - 1 \right| = \frac{x}{x+n} < \epsilon \text{ if } n > x\left(\frac{1}{\epsilon} - 1\right)$$

Let m denote the integer just greater than

$x\left(\frac{1}{\epsilon} - 1\right)$. clearly m increases

as x and $m \rightarrow \infty$ as $x \rightarrow \infty$

hence we can not find m such that for all $n > m$ and for all $x \geq 0$

$$|f_n(x) - f(x)| < \epsilon$$

Hence the convergence is not uniform in $[0, \infty]$ but if k is finite we can find a m such that

$$m > K\left(\frac{1}{\epsilon} - 1\right) > x\left(\frac{1}{\epsilon} - 1\right)$$

uniformly convergent in ~~$[0, \infty]$~~ ~~$[0, k]$~~ $[0, k]$